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INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4143–4150 PII: S0305-4470(02)28752-1

Duality and supersymmetric quantum mechanics

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Received 12 September 2001, in final form 31 January 2002
Published 26 April 2002
Online at stacks.iop.org/JPhysA/35/4143

Abstract

Given a SUSY potential $\Phi(x)$ in nonrelativistic quantum mechanics, a pair of potentials $V_\pm(x)$ can be constructed to describe the bosonic and fermionic sectors of the theory. We apply a form of electric–magnetic duality to such a pair of potentials and derive conditions that must be satisfied in order for the images of the original potentials under duality to still form a supersymmetric pair in one or two dimensions. These conditions serve as compatibility conditions between the supersymmetry and duality transformations.

PACS numbers: 03.65.−w, 11.30.Pb

1. Introduction

The term duality refers to an equivalence between two different theories or between two different parameter regions of the same theory; examples are duality between strong and weak coupling limits, high and low temperature regimes, perturbative and solitonic sectors, or between electric and magnetic sectors. The idea of duality has in recent years become a very important topic in QCD and supergravity, and is responsible for connecting the various string theory limits of M-theory. This work has a history that can be traced through investigations carried out in the 1970s [1, 2] back to Dirac's work on magnetic monopoles in the 1930s [3], but it exploded after the results of Seiberg and Witten [4] in the mid 1990s. (For reviews, see [5] for QCD; [6–8] for supersymmetric field theories; and [9] for M-theory. A number of articles concerning aspects of duality in both field theory and string theory can be found in [10].)

Given the importance of duality in field theory and string theory, it would appear to be interesting to study duality in the simpler context of nonrelativistic quantum mechanics, where the subject can be studied with many of the nonessential complications stripped away. Since the concept of duality seems to reach its fullest power when coupled with supersymmetry, it should be especially interesting to study it in the context of supersymmetric quantum mechanics. So far, only a few authors seem to have studied duality in quantum mechanics (for example, [11–13]), and even fewer in the supersymmetric case [14]. In this paper, we examine a duality transformation introduced in [12] and pursue the question of when this transformation is compatible with supersymmetry in specific models of supersymmetric quantum mechanics. More specifically, we apply the transformation to a set of supersymmetric pair potentials

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$V_\pm(x)$, producing a new set of potentials $\tilde{V}_\pm(u)$, and then derive conditions that must be satisfied in order for the new potentials to still form a supersymmetric pair. In section 3 we work in the context of the $N = 2$ Witten model in one dimension, and in section 4 we deal with a two-dimensional $N = 1$ model that includes the possibility of vector potentials.

2. Supersymmetric quantum mechanics

Here, we very briefly review supersymmetric quantum mechanics. N-extended supersymmetry (often denoted SUSY, for short) can be formulated in terms of a set of Hermitian generators $Q_i, i = 1, \ldots, N$. For even N, an equivalent formulation is often used with non-Hermitian generators $\tilde{Q}_i$ and $\tilde{Q}_i^\dagger$, where $\tilde{Q}_i = \frac{1}{\sqrt{2}}(Q_i + \mathrm{i}Q_{2i})$, for $i = 1, 2, \ldots, N/2$. Supersymmetry was first introduced into nonrelativistic quantum mechanics by Nicolai [15] and Witten [16]. Reviews can be found in [17, 18].

Supersymmetric quantum mechanics begins with a set of fermionic supercharge operators Q_i satisfying

$$\{Q_i, Q_j\} = H\delta_{ij}.$$

The Hamiltonian can be block diagonalized,

$$H = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

where the blocks correspond to the Hamiltonians for the fermionic and bosonic sectors. The most commonly studied model of supersymmetric quantum mechanics is the Witten model, in which the corresponding potential functions $V_\pm(x)$ can be written in terms of a single function $\Phi(x)$, known as the SUSY potential:

$$V_-(x) = \Phi^2(x) - \frac{\hbar}{\sqrt{2m}}\Phi'(x)$$

$$V_+(x) = \Phi^2(x) + \frac{\hbar}{\sqrt{2m}}\Phi'(x).$$

It should be noted that in this context the words 'bosonic' and 'fermionic' do not necessarily refer to particles of integer and half-integer spin, but refer to states with positive (bosonic) or negative (fermionic) eigenvalues of a Witten operator. A Witten operator W is any operator satisfying the conditions

$$[W, H] = 0 \qquad \{W, Q_i\} = 0 \qquad W^2 = 1.$$

(See [17] or [18] for more information on such operators.)

There are several equivalent criteria for determining whether the supersymmetry is good or whether it is spontaneously broken. For example, supersymmetry is broken if the ground state energy is nonzero. In terms of states, unbroken SUSY will require that the supercharges annihilate a normalizable ground state. Defining

$$f(x) = \int_{x_0}^{x} \Phi(x')\,\mathrm{d}x'$$

for any fixed, arbitrary point x_0, then in one dimension such a state exists for a potential defined on domain $a \leqslant x \leqslant b$ if $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$ both diverge and have opposite signs.

3. Duality and supersymmetry in one dimension

Here, we restrict ourselves to the simplest case of the $N = 2$ Witten model in one spatial dimension. The case of two dimensions will be examined in the next section. We will largely follow the approach taken in [12] for the case of nonsupersymmetric systems. There, the duality transformation we will use was shown to take electrically charged particles into dyons, i.e. particles carrying both electric and magnetic charges.

Begin with the pair of one-dimensional time-independent Schrödinger equations:

$$0 = (H_\pm - E)\psi_\pm(x) = \left(-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \Phi^2(x) \pm \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} - E\right)\psi_\pm(x).$$

Let

$$r = \frac{x^2}{L}$$

and define new wavefunctions

$$\psi_\pm(x) = r^\alpha \tilde{\psi}_\pm(r)$$

for some as-yet undetermined exponent α. The length scale L is present in order to make the dimensions work out correctly, and can be chosen to be any characteristic length scale of the problem. For example, L could be chosen to be $\frac{\hbar}{\sqrt{mE}}$. The Schrödinger equation now takes the form

$$\begin{aligned}0 = 4r^{\alpha+1}\Bigg\{&-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{\hbar^2}{8m}\left(\frac{2\alpha(2\alpha-1)}{r^2} + \frac{2(4\alpha+1)}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\\ &+ \frac{L}{4r}\left(\hat{\Phi}^2(r) \pm \hbar\sqrt{\frac{2r}{mL}}\frac{\mathrm{d}\hat{\Phi}(r)}{\mathrm{d}r} - E\right)\Bigg\}\tilde{\psi}_\pm(r)\end{aligned}$$

where we have defined

$$\hat{\Phi}(r) = \Phi(r(x)).$$

By choosing $\alpha = -\frac{1}{4}$, the first derivative term can be removed, leaving the equation

$$0 = \left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{3\hbar^2}{32mr^2} + \frac{L}{4r}\left(\hat{\Phi}^2(r) \pm \hbar\sqrt{\frac{2r}{mL}}\frac{\mathrm{d}\hat{\Phi}(r)}{\mathrm{d}r} - E\right)\right\}\tilde{\psi}_\pm(r).$$

We wish this again to be in the form of a Schrödinger equation

$$0 = \left\{-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \tilde{V}_\pm(r) - \tilde{E}\right\}\tilde{\psi}_\pm(r)$$

for some new set of energies $\tilde{E}$ and new pair potentials $\tilde{V}_\pm$ derived from some new SUSY potential $\tilde{\Phi}(r)$,

$$\tilde{V}_\pm(r) = \tilde{\Phi}^2(r) \pm \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}\tilde{\Phi}(r)}{\mathrm{d}r}.$$

This implies two conditions that must be satisfied:

(i) $$\frac{1}{2}\sqrt{\frac{L}{r}}\frac{\mathrm{d}\hat{\Phi}(r)}{\mathrm{d}r} = \frac{\mathrm{d}\tilde{\Phi}(r)}{\mathrm{d}r}$$

(ii) $$\frac{L}{4r}(\hat{\Phi}^2(r) - E) - \frac{3\hbar^2}{32mr^2} = \tilde{\Phi}^2(r) - \tilde{E}.$$

These two conditions can easily be combined into a single integral equation. Integrating relation (i) by parts,

$$\begin{aligned}\tilde{\Phi}(r) &= \int_{r_0}^{r}\frac{1}{2}\sqrt{\frac{L}{r'}}\frac{\mathrm{d}\hat{\Phi}(r')}{\mathrm{d}r'}\,\mathrm{d}r' + \hat{\Phi}(r_0)\\ &= -\int_{r_0}^{r}\left(-\frac{1}{4}\right)\sqrt{\frac{L}{(r')^3}}\hat{\Phi}(r')\,\mathrm{d}r' + \frac{1}{2}\sqrt{\frac{L}{r'}}\hat{\Phi}(r')\Big|_{r_0}^{r} + \hat{\Phi}(r_0)\end{aligned}$$

for any fixed r_0. Written in a slightly different form, this is

$$\text{(i}'\text{)}\qquad \tilde{\Phi}(r) = \frac{1}{4}\int \sqrt{\frac{L}{r^3}}\hat{\Phi}(r)\,\mathrm{d}r + \frac{1}{2}\sqrt{\frac{L}{r}}\hat{\Phi}(r) + K$$

where now the integral is indefinite and K is an integration constant. Also, rearranging (ii), we have

$$\text{(ii}'\text{)}\qquad \tilde{\Phi}(r) = \pm\sqrt{\tilde{E} - \frac{3\hbar^2}{32mr^2} + \frac{L}{4r}(\hat{\Phi}^2(r) - E)}.$$

Comparing the last two equations, we have the desired necessary compatibility condition:

$$\frac{1}{4}\int \sqrt{\frac{L}{r^3}}\hat{\Phi}(r)\,\mathrm{d}r + \frac{1}{2}\sqrt{\frac{L}{r}}\hat{\Phi}(r) + K = \pm\sqrt{\tilde{E} - \frac{3\hbar^2}{32mr^2} + \frac{L}{4r}(\hat{\Phi}^2(r) - E)}. \tag{3.1}$$

It is only systems with SUSY potentials satisfying this condition that will again be mapped into supersymmetric systems by the duality transformation.

Instead of an integral equation, this result can be written in the form of an equivalent differential equation. Starting from conditions (i) and (ii), it is straightforward to show that

$$\hat{\Phi}'(r) = \pm\left[\frac{\frac{3\hbar^2}{8mr^2} - \frac{L}{2r}(\hat{\Phi}^2(r) - E)}{\sqrt{(\hat{\Phi}^2(r) - E) - \frac{3\hbar^2}{8mrL} + \frac{4r}{L}\tilde{E}} \mp \hat{\Phi}(r)}\right].$$

This could also be written as a differential equation for $\tilde{\Phi}$ instead of $\hat{\Phi}$. In this differential context, the constant K again appears as an integration constant; specifically, it fixes the asymptotic value of $\tilde{\Phi}$:

$$K = \lim_{r\to\infty}\tilde{\Phi}(r).$$

To get some idea of the sorts of systems that satisfy these conditions we can try some trial forms for Φ and see what constraints are forced onto the parameters. For example, consider the trial solution

$$\hat{\Phi}(r) = Cr^{\beta}$$

for unknown constants C and β. Plugging this into equation (3.1), it is easy to see that (3.1) can only be satisfied if

$$K^2 - \tilde{E} = 0$$
$$\beta = -\tfrac{1}{2}$$

which in turn implies that

$$C^2 = \frac{\hbar^2}{2mL}$$
$$K = -\frac{E\sqrt{L}}{2C} = \pm\sqrt{\frac{m}{2}}\left(\frac{EL}{\hbar}\right).$$

The energies of the two dual systems are therefore related by

$$\tilde{E} = L\left(\frac{E}{2C}\right)^2.$$

The dual SUSY potentials are

$$\Phi(x) = \frac{C}{x}\sqrt{L}$$
$$\tilde{\Phi}(r) = \frac{C\sqrt{L}}{4r} + K.$$

The scalar potentials are given by

$$V_{\pm}(x) = \frac{C(C \mp C)L}{x^2} = \begin{cases} \dfrac{\hbar^2}{mx^2} & \text{for} \quad \tilde{V}_{-} \\ 0 & \text{for} \quad \tilde{V}_{+} \end{cases}$$

$$\begin{aligned} \tilde{V}_{\pm}(r) &= \frac{C(C \mp 4C)L}{16r^2} + \left(\frac{CK\sqrt{L}}{2r} + K^2\right) \\ &= \frac{\hbar^2(1 \mp 4)}{32mr^2} - \frac{EL}{4r} + \frac{mE^2L^2}{2\hbar^2} \\ &= \begin{cases} +\dfrac{5\hbar^2}{32mr^2} - \dfrac{EL}{4r} + \dfrac{mE^2L^2}{2\hbar^2} & \text{for} \quad \tilde{V}_{-} \\ -\dfrac{3\hbar^2}{32mr^2} - \dfrac{EL}{4r} + \dfrac{mE^2L^2}{2\hbar^2} & \text{for} \quad \tilde{V}_{+} \end{cases}. \end{aligned}$$

Note that all of the potentials turn out to be independent of the choice of sign for C.

Several things may be noted about these results. For example, the potentials $\tilde{V}_{\pm}(r)$ form a shape-invariant pair. Shape invariance means that there is a mapping from the parameters $\{a_i\}$ of one potential to the parameters $\{b_i\}$ of the other potential such that

$$V_{+}(a_i, \boldsymbol{r}) = V_{-}(b_i, \boldsymbol{r}) + R(b_i)$$

where $R(a_i)$ is independent of $\boldsymbol{r}$. Shape invariance is a sufficient condition to guarantee exact solvability [17, 18]. Formally, these two potentials represent the radial parts (with centripetal barrier) of two Coulomb potentials:

$$V_{\text{Coulomb}}(r) = -\frac{\alpha}{r} + \frac{\hbar^2\eta(\eta \pm 1)}{2mr^2} + \frac{em\alpha^2}{2\eta^2\hbar^2}.$$

Here, α is the fine-structure constant, $\eta = l + 1$, and l is the angular momentum quantum number. These two Coulomb problems have different values of angular momentum, but the same charge. However, they are nonphysical in the sense that they have negative and fractional angular momentum quantum numbers: $l = -\frac{3}{4}$ and $l = -\frac{5}{4}$. The *physical* Coulomb problem (with positive, integer-valued quantum number) is known to have a hidden supersymmetry [19], but the formalism here implies that the supersymmetry will not be preserved by the duality transformation.

Second, the original SUSY potential $\Phi(x)$ describes a system with unbroken SUSY on the full real-number line, while its dual $\tilde{\Phi}(r)$ describes a system with unbroken supersymmetry defined on the half-line $r \geqslant 0$. Thus, in this example at least, the duality transformation preserves the 'goodness' of supersymmetry.

4. Duality and supersymmetry in two dimensions

In going from one to two spatial dimensions, we introduce the possibility of vector potentials appearing. Suppose we have a vector potential $\boldsymbol{A}(\boldsymbol{x})$ and a SUSY potential $\Phi(\boldsymbol{x})$ in two dimensions. In the simplest case, we can introduce a single Grassmann supercharge according to

$$\sqrt{2}Q = \frac{1}{\sqrt{2m}}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)\sigma + \Phi\sigma_3.$$

Here, $\boldsymbol{p}$ and $\boldsymbol{A}$ are 2-vectors in the x–y plane. The two-dimensional 'magnetic' field $B = \partial_x A_y - \partial_y A_x$ will be a scalar function; if there were a third dimension it would be the z-component of a 3-vector magnetic field. The Hamiltonian is found to be

$$H = 2Q^2 = \begin{pmatrix} \frac{1}{2m}\left(\mathrm{i}\hbar\nabla_{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}\right)^2 - \frac{e\hbar}{2mc}B + \Phi^2 & \frac{\hbar}{\sqrt{2m}}[(\partial_2\Phi) + \mathrm{i}(\partial_1\Phi)] \\ \frac{\hbar}{\sqrt{2m}}[(\partial_2\Phi) - \mathrm{i}(\partial_1\Phi)] & \frac{1}{2m}\left(\mathrm{i}\hbar\nabla_{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}\right)^2 + \frac{e\hbar}{2mc}B + \Phi^2 \end{pmatrix}.$$

Block-diagonalizing, we find the pair of Hamiltonians:

$$H_\pm = \frac{1}{2m}\left(\mathrm{i}\hbar\nabla_{\boldsymbol{x}} + \frac{e}{c}\boldsymbol{A}\right)^2 + \Phi^2 \pm \frac{\hbar}{2m}\sqrt{\frac{e^2}{c^2}B^2 + 2m\left[(\partial_1\Phi)^2 + (\partial_2\Phi)^2\right]}.$$

Writing down the Schrödinger equations for these Hamiltonians, and then going to polar coordinates

$$u^2 = x^2 + y^2 \qquad \phi = \tan^{-1}\frac{y}{x}$$

we arrive at

$$0 = \left\{ -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial u^2} + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{u^2}\frac{\partial^2}{\partial\phi^2}\right) + \left(\Phi_p^2(u,\phi) - E\right) \right.$$
$$+ \frac{e^2}{2mc^2}\boldsymbol{A}_{\boldsymbol{p}}^2 + \frac{\mathrm{i}\hbar e}{2mc}(\nabla_{\boldsymbol{x}}\cdot\boldsymbol{A}_{\boldsymbol{p}}) + \frac{\mathrm{i}\hbar e}{mc}\boldsymbol{A}_{\boldsymbol{p}}\cdot\nabla_{\boldsymbol{x}}$$
$$\left. \pm \frac{\hbar}{\sqrt{2m}}\left[\frac{e^2}{2mc^2}B_p^2 + \left(\frac{\partial\Phi_p(u,\phi)}{\partial u}\right)^2 + \frac{1}{u^2}\left(\frac{\partial\Phi_p(u,\phi)}{\partial\phi}\right)^2\right]^{1/2} \right\}\psi_\pm(u,\phi)$$

where the subscript p attached to a variable denotes that the variable is now written as a function of the polar coordinates.

Now apply the two-dimensional version of the duality transformation:

$$r = \frac{u^2}{L} \qquad \theta = 2\phi.$$

The length scale L could again be chosen to be $L = \frac{\hbar c}{\sqrt{mE}}$. (In the case where B is a constant, uniform field we could alternatively choose either $L = \sqrt{\frac{\hbar c}{eB}}$ or $L = \frac{mc}{eB}$.) In (r,θ) coordinates, the Schrödinger equation becomes

$$0 = \left\{ -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) + \frac{L}{4r}(\hat{\Phi}^2(r,\theta) - E) \right.$$
$$+ \frac{e^2}{2mc^2}\hat{\boldsymbol{A}}^2 + \frac{\mathrm{i}\hbar e}{2mc}[(\nabla_{\boldsymbol{r}}\cdot\hat{\boldsymbol{A}}) + 2\hat{\boldsymbol{A}}\cdot\nabla_{\boldsymbol{r}}]$$
$$\left. \pm \frac{\hbar}{\sqrt{2m}}\left[\frac{e^2}{2mc^2}\hat{B}^2 + \frac{L}{4r}\left(\frac{\partial\hat{\Phi}(r,\theta)}{\partial r}\right)^2 + \frac{L}{4r^3}\left(\frac{\partial\hat{\Phi}(r,\theta)}{\partial\theta}\right)^2\right]^{1/2} \right\}\psi_\pm(r,\theta)$$

where

$$\hat{\Phi}(r,\theta) = \Phi_p(u(r,\theta),\phi(r,\theta))$$
$$\hat{\boldsymbol{A}}(r,\theta) = \boldsymbol{A}_p(u(r,\theta),\phi(r,\theta))$$
$$\hat{B}(r,\theta) = \frac{L}{4r}B_p(u(r,\theta),\phi(r,\theta)).$$

Because of the doubling of the angular variable, the wavefunction is now defined on a two-sheeted Riemann surface. There are two independent solutions for each r and θ; one is unchanged by a 2π rotation, the other picks up a minus sign. So, following [12], we make the substitution

$$\psi_\pm(r,\theta) = \mathrm{e}^{\mathrm{i}s\theta}\bar{\psi}_\pm^{(s)}(r,\theta)$$

with $s = 0$ or $s = \frac{1}{2}$. $\bar{\psi}_\pm^{(s)}$ has the usual periodicity

$$\bar{\psi}_\pm^{(s)}(r,\theta+2\pi) = \bar{\psi}_\pm^{(s)}(r,\theta)$$

for both values of s.

The Schrödinger equations in terms of $\bar{\psi}_\pm^{(s)}$ are now defined on ordinary 2-space, and are given by

$$0 = \left\{-\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} - \mathrm{i}s\right)^2\right] + \frac{L}{4r}(\hat{\Phi}^2(r,\theta) - E) + \frac{e^2}{2mc^2}\hat{\boldsymbol{A}}^2 - \frac{se\hbar}{mc}\frac{\hat{A}_\theta}{r^2}\right.$$
$$+\frac{\mathrm{i}\hbar e}{2mc}((\boldsymbol{\nabla}_r\cdot\hat{\boldsymbol{A}}) + 2\hat{\boldsymbol{A}}\cdot\boldsymbol{\nabla}_r) \pm \frac{\hbar}{\sqrt{2m}}\left[\frac{e^2}{2mc^2}\hat{B}^2 + \frac{L}{4r}\left(\frac{\partial\hat{\Phi}(r,\theta)}{\partial r}\right)^2\right.$$
$$\left.\left.+\frac{L}{4r^3}\left(\frac{\partial\hat{\Phi}(r,\theta)}{\partial\theta}\right)^2\right]^{1/2}\right\}\bar{\psi}_\pm^{(s)}(r,\theta).$$

For each s, define new vector potentials on the (r_1, r_2)-plane:

$$\boldsymbol{A}_s^{(g)} = \frac{g_s}{r^2}(r_2, -r_1)$$

where the two-dimensional 'electric' and 'magnetic' charges e and g_s are related by

$$g_s = \frac{\hbar cs}{e}.$$

This potential produces a vanishing magnetic field $B_s^{(g)}$ except at $r = 0$. Note that even if the vector potential is initially zero, a nonzero potential will be introduced by the duality transformation. In terms of this new field, the previous equation becomes

$$0 = \left\{-\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}e}{\hbar c}\left(\boldsymbol{A}_s^{(g)} + \hat{\boldsymbol{A}}\right)\right)^2 + \frac{L}{4r}(\hat{\Phi}^2 - E)\right.$$
$$\left.\pm\frac{\hbar}{\sqrt{2m}}\left[\frac{e^2}{2mc^2}\hat{B}^2 + \frac{L}{4r}\left(\frac{\partial\hat{\Phi}}{\partial r}\right)^2 + \frac{L}{4r^3}\left(\frac{\partial\hat{\Phi}}{\partial\theta}\right)^2\right]^{1/2}\right\}\psi_\pm^{(s)}(r,\theta).$$

We wish to be able to put this back into the Schrödinger form:

$$0 = \left\{-\frac{\hbar^2}{2m}\left(\nabla - \frac{\mathrm{i}e}{\hbar c}\tilde{A}_s\right)^2 + (\tilde{\Phi}^2 - \tilde{E})\right.$$
$$\left.\pm\frac{\hbar}{\sqrt{2m}}\left[\frac{e^2}{2mc^2}\tilde{B}_s^2 + \left(\frac{\partial\tilde{\Phi}_s}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\tilde{\Phi}_s}{\partial\theta}\right)^2\right]^{1/2}\right\}\bar{\psi}^{(s)}(r,\theta).$$

This implies conditions analogous to those in section 3:

(i) $\tilde{\boldsymbol{A}}_s = \hat{\boldsymbol{A}} + \boldsymbol{A}_s^{(g)}$

(ii) $\frac{L}{4r}(\hat{\Phi}^2 - E) = \tilde{\Phi}_s^2 - \tilde{E}_s$

(iii) $\frac{e^2}{2mc^2}\hat{B}^2 + \frac{L}{4r}\left(\frac{\partial\hat{\Phi}}{\partial r}\right)^2 + \frac{L}{4r^3}\left(\frac{\partial\hat{\Phi}}{\partial\theta}\right)^2 = \frac{e^2}{2mc^2}\tilde{B}_s^2 + \left(\frac{\partial\tilde{\Phi}_s}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\tilde{\Phi}_s}{\partial\theta}\right)^2.$

As before, these can be combined into a single integral or differential equation. Note that the first equation implies that

$$\tilde{B}_s = \hat{B} + 2\pi g_s \delta(\boldsymbol{r})$$

so that $\tilde{B}_s$ could be eliminated from (iii).

5. Conclusion

It has been shown here that in quantum mechanics duality transformations and supersymmetry can coexist, but only if certain compatibility conditions are satisfied. The generalization to higher dimension or larger N is straightforward. These relatively simple conditions are very restrictive in one spatial dimension; as the number of dimensions increases, they become more complicated but less restrictive due to the appearance of additional degrees of freedom.

Once the issue of duality and supersymmetry in quantum mechanics has been raised, many new questions open up, such as the compatibility of duality with shape invariance, or the compatibility of supersymmetry with more general notions of duality than the one considered here. These questions are left for future study.

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